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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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PERIODIC FORESTS WHOSE LARGEST CLEARINGS ARE OF SIZE 3

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CONTENTS

Miller has observed that there are a finite number of periodic forests whose largest clearings are of size 1 or 2, and an infinite number whose largest clearings are of size 4. In this note the basic theory of periodic forests is outlined, and the number of periodic forests whose largest clearings are of size 3 is examined. There are 12 such forests; their corresponding tessellations are sketched.

1. Introduction

The study of periodic forests of stunted trees, initiated by J. C. P. Miller (see 1970, preceding paper) was developed by him at a recent symposium.† In the course of his paper (Miller 1968) he noted that there is just one periodic forest whose largest clearings are of size 1, just one (apart from reflexion) periodic forest whose largest clearings are of size 2, and an infinity of periodic forests whose largest clearings are of size 4; he left as an open question the problem of determining whether the number of periodic forests whose largest clearings are of size 3 is finite or infinite. In this note it is shown that there are just 12 distinct periodic forests whose largest clearings are of size 3; and their corresponding tessellations are sketched.

The process used is, effectively, one of enumeration of cases; as such it is most inelegant, and the writer trusts that it will be rapidly replaced by a shorter and more attractive proof. The result, however, is of interest; both in itself and in filling an otherwise awkward gap in the theory.

2. Forests of stunted trees

We are concerned with an infinite background of *nodes*, situated at the vertices of a plane tessellation of equilateral triangles of unit side.

We consider as the *ground* a row of nodes at unit distance apart, and arbitrarily label the

† I.B.M. Symposium on Utilization of Computers in Mathematical Research: Blaricum 1966.

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nodes of this row as *live* or *vacant*. We think of each successive row of nodes on one side of the ground and parallel to it as being at successively higher *levels*. Any live node, at any level (including the ground), can give rise to a *branch* to one or other or both of the two nearest nodes on the next higher level. Such a branch occurs, on either side, if and only if it is not *stunted* (i.e. prohibited) by the presence of an immediately adjacent live node on the same side. Nodes on the higher level that have been reached by branches are then labelled as live; nodes not so reached are labelled as vacant. Thus the pattern of live and vacant nodes on the ground uniquely determines the pattern of live and vacant nodes at each successive level above the ground. (The pattern of branches constitutes a *forest*; each live node on the ground can be thought of as a *root*; roots give rise to *trees* unless their growth is stunted, and trees branch at their live nodes at higher levels unless such branching is inhibited by the presence of immediately adjacent live nodes.)

3. Periodic forests

We now restrict our attention to the situation in which the pattern of live nodes on the ground is periodic. Let the minimum period be n; any row of nodes of minimum period n leads to a row of nodes of minimum period n or $\frac{1}{2}n$ at the next higher level (Miller 1970), and since the number of distinct rows of minimum period n is bounded it follows that repetition must eventually occur.

It is not necessarily the case that the pattern of the ground row will recur at a higher level; in general there will be a number of unrepeated rows before the first row is reached that is repeated at a higher level. We refer to the minimum period of this (or any higher) row as the *ground-period* of the forest, and the number of levels through which we have to ascend from a row to its first repetition as the *row-period*.

That part of the forest below the first repeating row is now removed and replaced by an extension downwards of the established repeating pattern of live and vacant nodes. The resultant forest is a *periodic forest*. (It will be noted that in a periodic forest no row is distinguished as the ground; the 'ground' has been dismissed downwards to infinity, and the trees of the forest are of infinite extent downwards as well as upwards. The concept of a ground direction, parallel to the rows, is still valid, however.)

4. Triads of periodic forests

A forest consists of branches; each branch joins a pair of adjacent live nodes on different levels. The live nodes constitute the *background* of the forest. We refer to as *permissible* any pattern of live nodes that constitutes the background of a periodic forest. Miller (1970) has shown that if we rotate a permissible pattern through $\frac{2}{3}\pi$ the resultant pattern is still permissible. Consequently, a permissible pattern is permissible from whichever of the three possible aspects it is viewed, and corresponds to a *triad* of periodic forests, one for each of the three possible ground directions. These three forests may all be congruent; two may be congruent and the third distinct; or all may be distinct.

5. Forest-tessellations

Given a permissible pattern of live nodes we can choose one direction as the ground direction and join all pairs of adjacent nodes in the other two directions to give a periodic forest.

A *forest-tessellation* is formed by joining pairs of adjacent live nodes in all three directions; it can alternatively be thought of as the super-position of the three forests of a triad.

A tessellation can be divided (in a variety of ways) into congruent *cells*. The smallest such cells have $2C$ unit triangles, where C is the product of the ground-period and row-period of any one of the three forests of the tesselation (Miller 1968).

Miller categorizes tessellations according to the congruence and reflexive properties of their three forests:

(*a*)	U-tessellation	three distinct unsymmetric forests
(*b*)	R-tessellation	one reflexive forest, and a pair of mirror-image unsymmetric forests
(*c*)	S-tessellation	three identical unsymmetric forests
(*d*)	T-tessellation	three identical reflexive forests

6. Clearings

A forest-tesselation is readily seen to consist of a plane-filling of triangles (of unit side), regular hexagons (of unit side), and alternate-sided hexagons (with three unit sides) of various sizes, all with triangular symmetry. The vacant nodes occur as triangular clusters, or *clearings*, surrounded by connected sets of adjacent live nodes.

The *size of a clearing* is defined as the number of vacant nodes in the side of the triangle of vacant nodes constituting the clearing. Thus a clearing of size 1 gives a regular hexagon in the tessellation; a clearing of size n gives a hexagon whose sides are alternately 1 and n units in length. For completeness we think of a triangle of three live nodes in the tessellation as surrounding a clearing of size 0.

7. Periodic forests whose largest clearings are of size less than 3

Miller has shown that there is a unique periodic forest having largest clearings of size 1 and a unique (except for reflexion) periodic forest having largest clearings of size 2.

The following demonstration of the second of these theorems is given in order to illustrate the method used in approaching the problem of periodic forests whose largest clearings are of size 3.

We label live nodes as ‘ × ’ and vacant nodes as ‘ · ’.

A periodic forest whose largest clearings are of size 2 involves, somewhere, a row segment

$$\times \cdot \cdot \times$$

This implies at the next lower level either

$$\cdot \times \times \times \cdot \quad \text{or} \quad \times \cdot \cdot \cdot \times$$

giving either

$$\begin{array}{c} \times \cdot \cdot \times \\ \cdot \times \times \times \cdot \end{array} \quad \text{or} \quad \begin{array}{c} \times \cdot \cdot \times \\ \times \cdot \cdot \cdot \times \end{array}$$

The second alternative can be ignored, since it implies the existence of a clearing of size 3 (or more). The first alternative implies at the next lower level either

× × · × · · or · · × · × ×,

giving either

```
 ×  ·  ·  ×         or        ×  ·  ·  ×
·  ×  ×  ×  ·                ·  ×  ×  ×  ·
×  ×  ·  ×  ·  ·          ·  ·  ×  ·  ×  ×
```

Only the first of these two alternatives need be considered further, since the second is a reflexion of the first. Extending the first successively to lower levels, it is clear by the preceding argument that there is only one viable possibility at each stage, and we obtain

```
      ×  ×  ·  ×  ·  ·
   ·  ×  ·  ·  ×  ×  ×
·  ·  ×  ×  ×  ·  ×  ·
×  ×  ×  ·  ×  ·  ·  ×  ×
```

The row-segment last added contains as a subset

× × · × · ·

and we now know that this can have below it only the pattern shown. Hence the downward extension of the pattern is unique.

The forest is unsymmetric; it has ground-period 7 and row period 1. The corresponding tesselation is of type S; its minimum cell contains one clearing of size 0 and one clearing of size 2.

8. Periodic forests whose largest clearings are of size 3

A periodic forest whose largest clearings are of size 3 involves, somewhere, a row-segment

× · · · ×

Downward development of this row-segment generates—as might be expected—an almost unmanageably large number of possibilities. In investigating these it is clearly a matter of practical convenience to examine particular sectors separately; it is also a matter of logical necessity (if the principles of §7 are followed), since a sector can only be fully disposed of if its initial row-segment leads to previously disposed of sectors, to dead-ends, and to at most one row-segment that contains the initial row-segment as a subset.

Determination of the initial row-segments of the various sectors—and the order in which they should be examined—was largely a matter of trial and error. The argument relating to the first of these is now given.

8.1. We consider, first, the row-segment

· · × × · × × · × · × × × · · × × × (1)

This has, at first sight, two possible downward continuations:

× × × · × × · × × · · × · × × × · × · (2)

and · · · × · · × · · × × · × · · · · × · × (2′)

The two possible downward continuations of (2′) are

· · · · × × × · · · × · · × × × × · · ×

and

× × × × · · · × × × · × × · · · · × × ·

each of which contains four adjacent vacant nodes. Hence (2′) has no viable downward continuation, and we can take (2) as the only possible downward continuation of (1). We continue this process until we arrive at

· · × · × · · × × · × × · × · × × × · · × × × · × × · × × (3)

as the only possible (eleven levels down) downwards continuation of (1), and we note that (3) contains (1) as a subset. Hence the continued viable downward continuation of (1) is unique. (We have not shown that it does not terminate at some later stage; but for the present it is sufficient to show that it leads to *at most* one periodic forest.)

We label the sequence of row-segments developable upwards from (3) with the symbol {10, 7, 4, 3}, the significance of which is given later.

8.2. The full examination is summarized in table 1. It can be shown that the row-segment [*n*] specified in the *n*th row and second column of the table leads only to the earlier row-segments specified in the third column and/or to a row-segment that contains the initial row-segment as a subset (in which case an entry of the form {*a*, *b*, *c*, *d*} is made in the fourth column of the table). It follows that the only viable downward continuations of · · · are those indicated in the fourth column of table 1.

Table 1

[1]	· · × × · × × · × · × × × · · × × ×	—	{10, 7, 4, 3}
[2]	· · · × × · · · × · · × · · ·	—	{6, 4, 0, 3}
[3]	× × · × · × × · · × · · × ×	—	{9, 6, 1, 4}
[4]	× · · × · · · × × · ·	[1], [2], [3]	—
[5]	× × · × · × × · · × · ·	[4]	{2, 1, 0, 1}
[6]	· × · · × × · × · · × × × · × × · · · × · × · · · ×	—	{61, 21, 19, 21}
[7]	· × · × × × · × · · × · · ×	[1], [6]	{36, 15, 12, 12}
[8]	× · · × · · · × ×	[4], [5], [7]	{61, 21, 19, 21}
[9]	· · · × × × · · × · ×	[8]	{13, 9, 7, 3}
[10]	· · × · · × · × × ×	[8], [9]	{3, 3, 1, 1}
[11]	× × · · · × × × · × · × ×	[8], [10]	{10, 7, 4, 3}
[12]	× · × × · · × ·	[8], [10], [11]	{10, 3, 4, 3}
[13]	× × × · · × × ×	[12]	{2, 0, 0, 1}
[14]	· · ·	[12], [13]	{2, 1, 0, 1}

8.3. Further development of the thirteen repetitive continuations that have been obtained leads to ten distinct tessellations: two of these are of type T, six of type S, and two of type R, so that (ignoring reflexions) there are just twelve distinct periodic forests whose largest clearings are of size 3.

The tessellations are shown on pages 119–121; in table 2 they are listed with their cell composition and the ground and row periods of their constituent forests. The symbol {*a*, *b*, *c*, *d*} previously attached to each repetitive continuation found in the analysis identifies the tessellation (by its number of clearings of size 0, 1, 2, 3 in its minimum cell) to which the continuation belongs.

8.4. The system of colouring used in the figures is of no special significance; it was felt that the arrangements adopted by the different sized clearings had some aesthetic value, and that they stood out more clearly in colour than could be the case with black-and-white sketches.

Table 2. Complete list of tessellations whose largest clearings are of size 3

number of clearings, in minimum cell, of size				constituent forests		
0	1	2	3	ground-period	row-period	tesselation type
2	0	0	1	6	2	T
2	1	0	1	{5, 15}	{3, 1}	R
3	3	1	1	14	2	S
6	4	0	3	12	4	T
10	3	4	3	73	1	S
9	6	1	4	73	1	S
10	7	4	3	{17, 85}	{5, 1}	R
13	9	7	3	28	4	S
36	15	12	12	273	1	S
61	21	19	21	56	8	S

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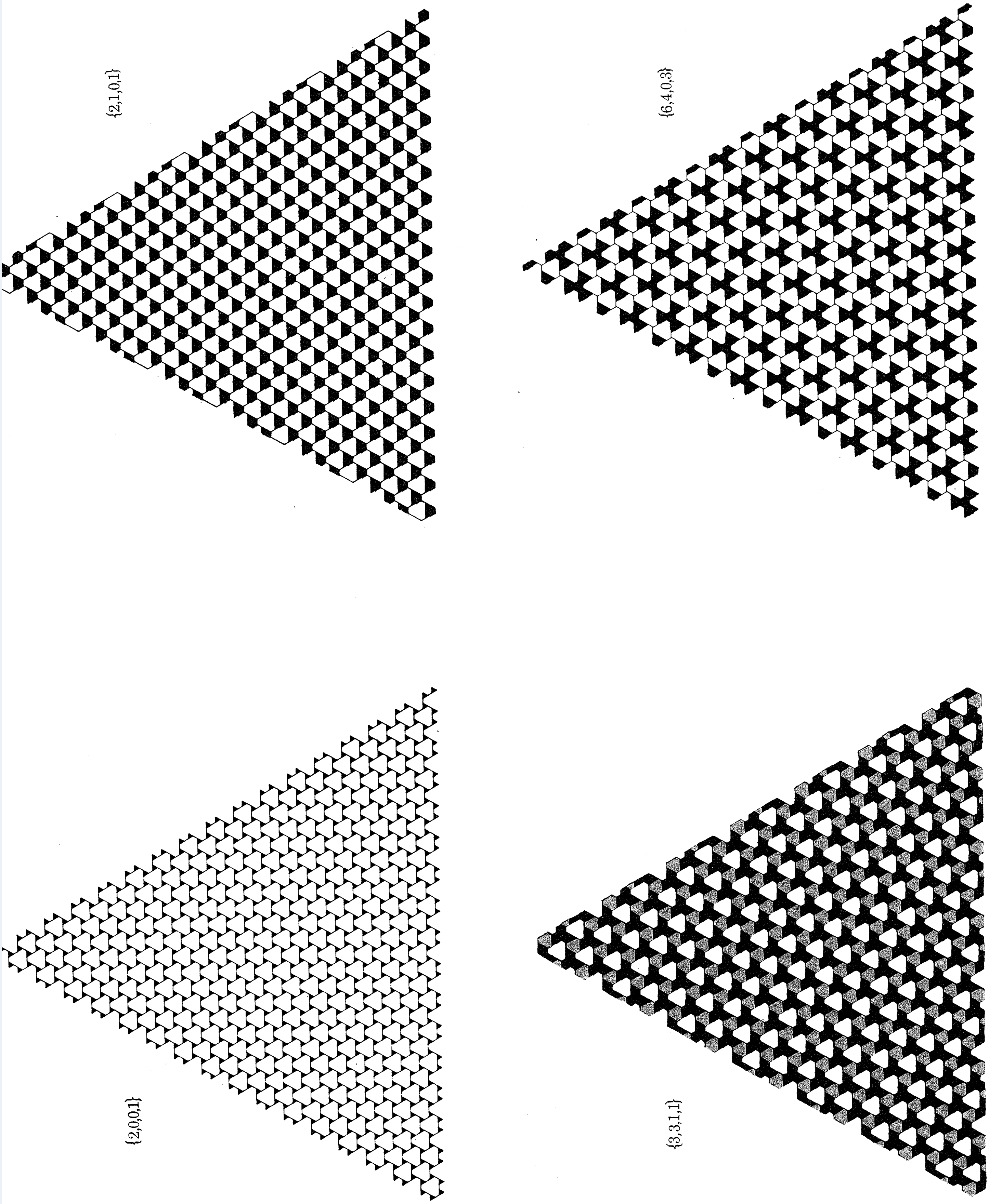
Miller, J. C. P. 1968 Periodic forests of stunted trees: the identification of distinct forests, *Computers in mathematical research* (editors R. F. Churchhouse and J.-C. Herz). Amsterdam: North Holland Publishing Co.

Miller, J. C. P. 1970 Periodic forests of stunted trees. *Phil. Trans. Roy. Soc. Lond.* A **266**, 63–111.



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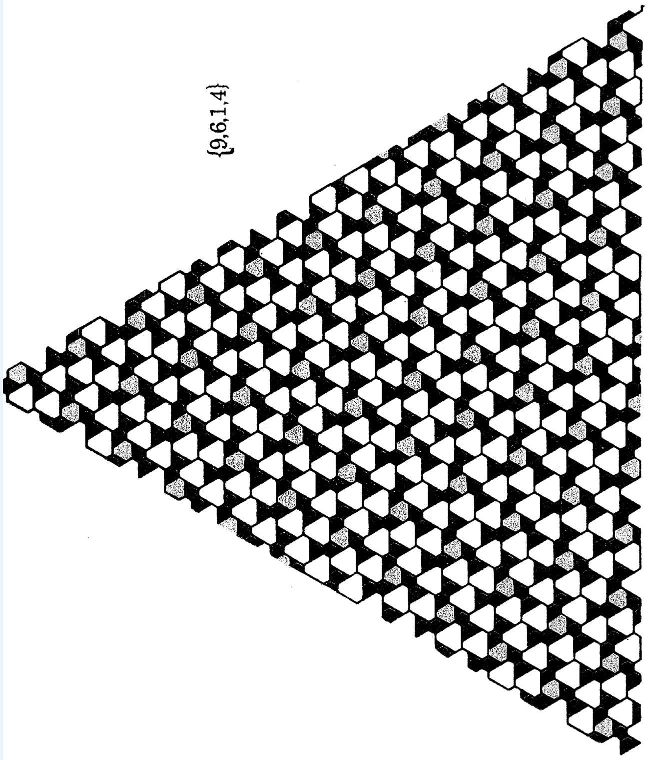
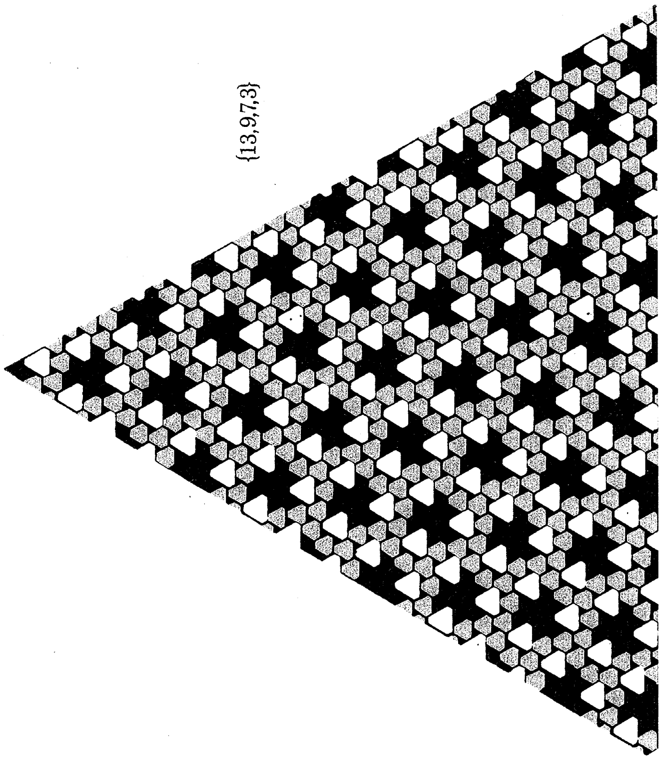
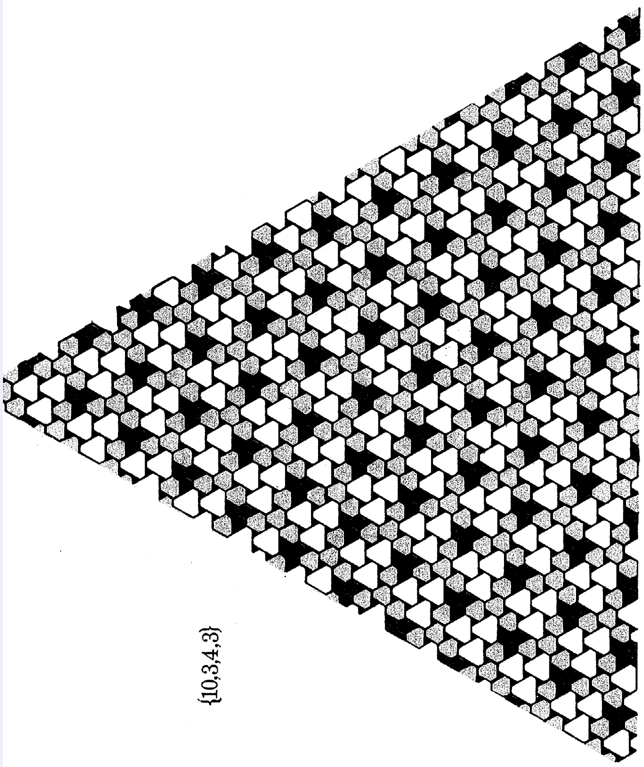


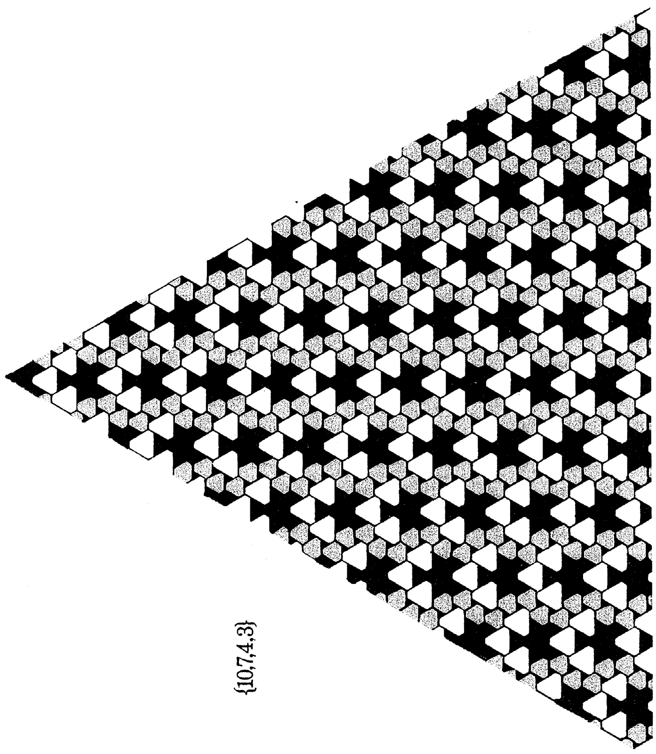
{9,6,1,4}



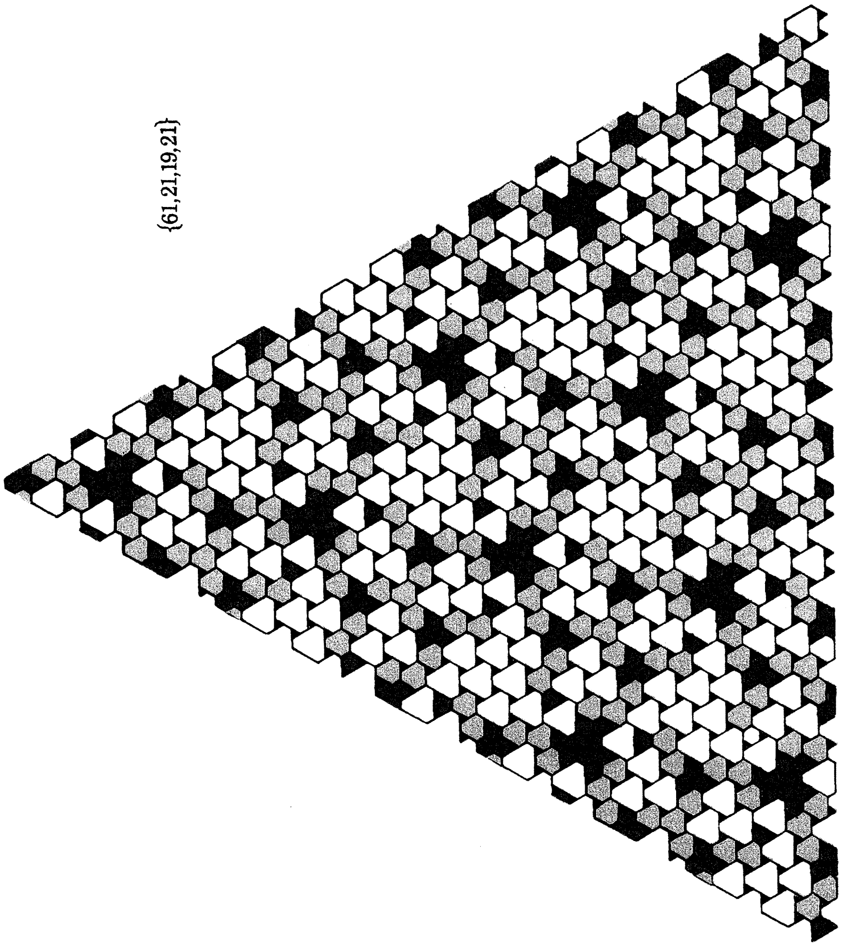
{13,9,7,3}



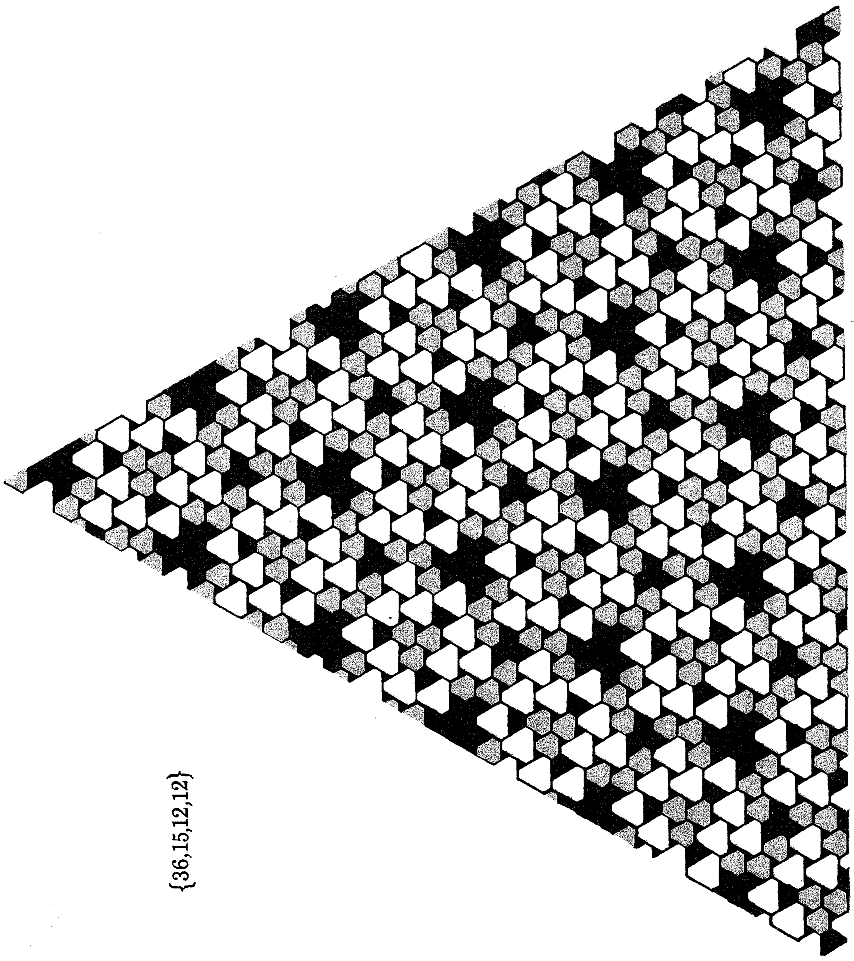
{10,3,4,3}



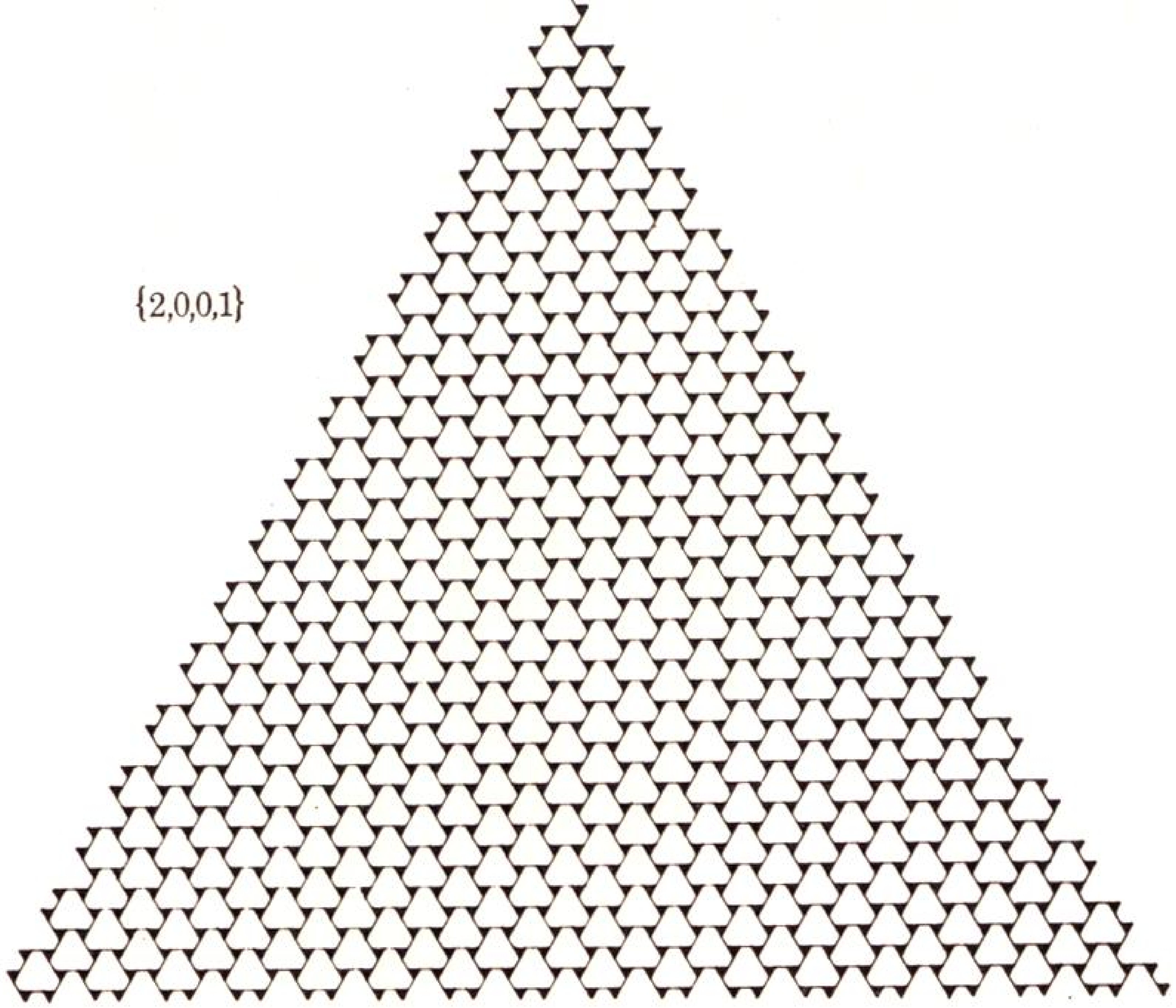
{10,7,4,3}



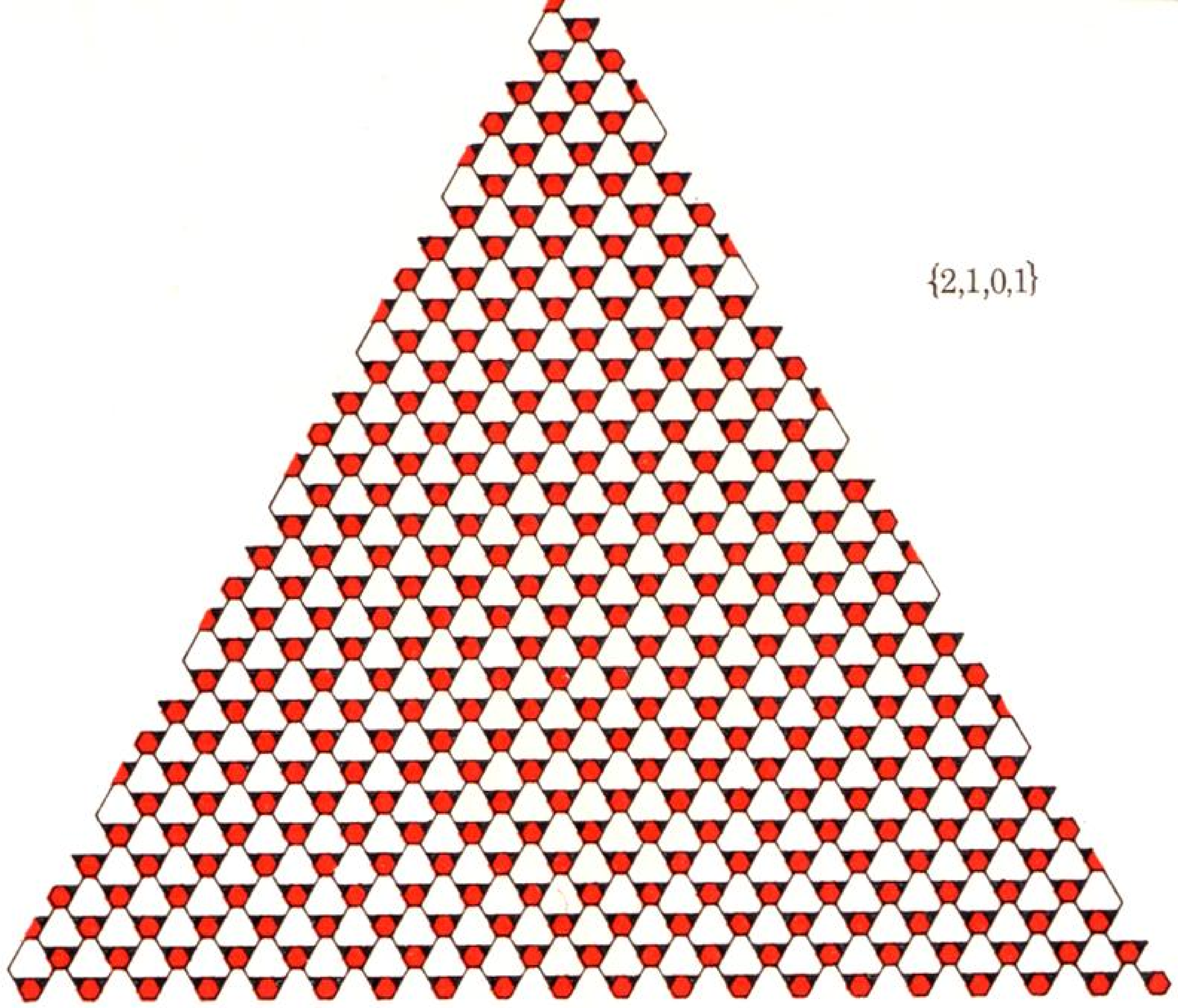
{61,21,19,21}



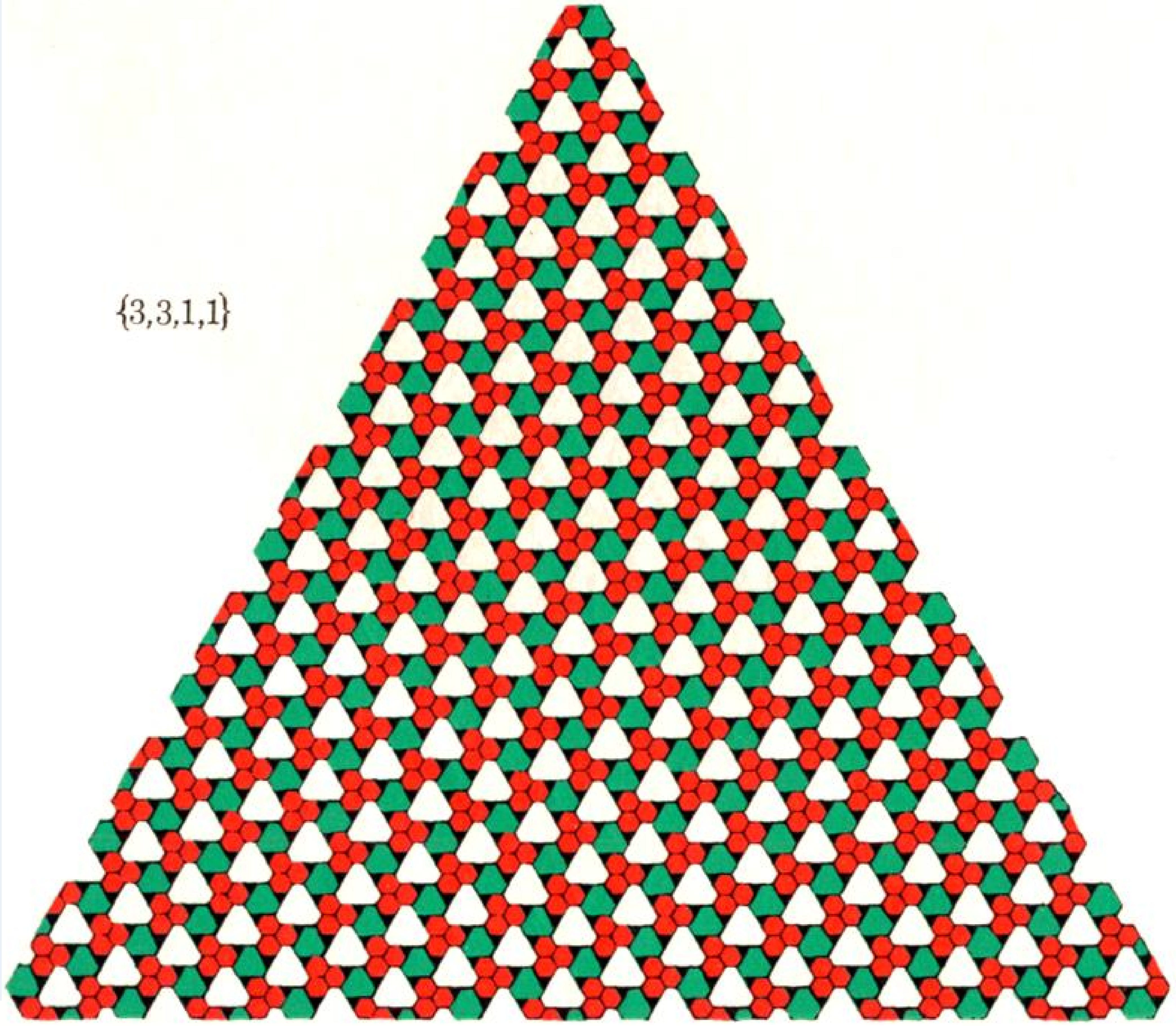
{36,15,12,12}



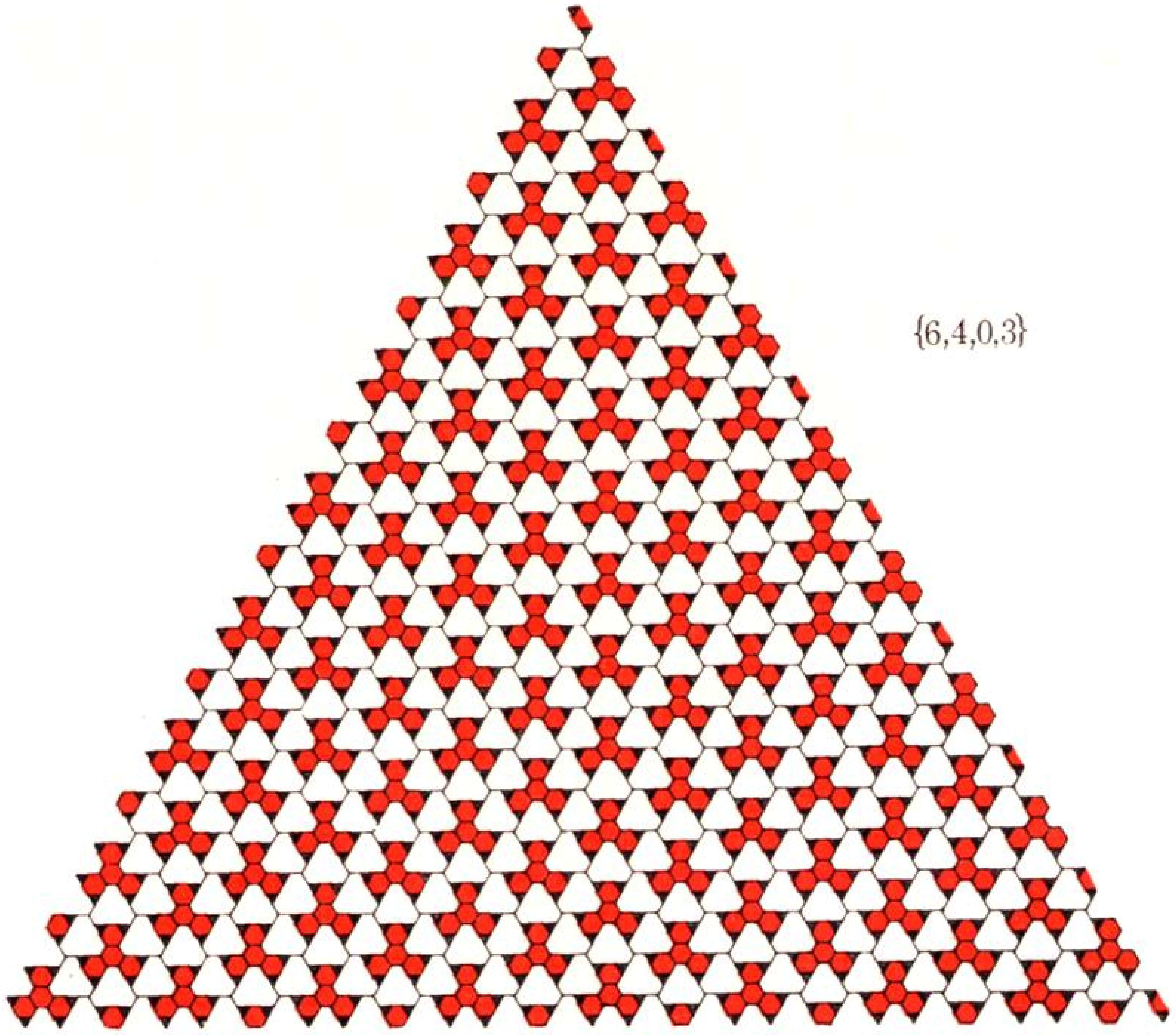
{2,0,0,1}



{2,1,0,1}

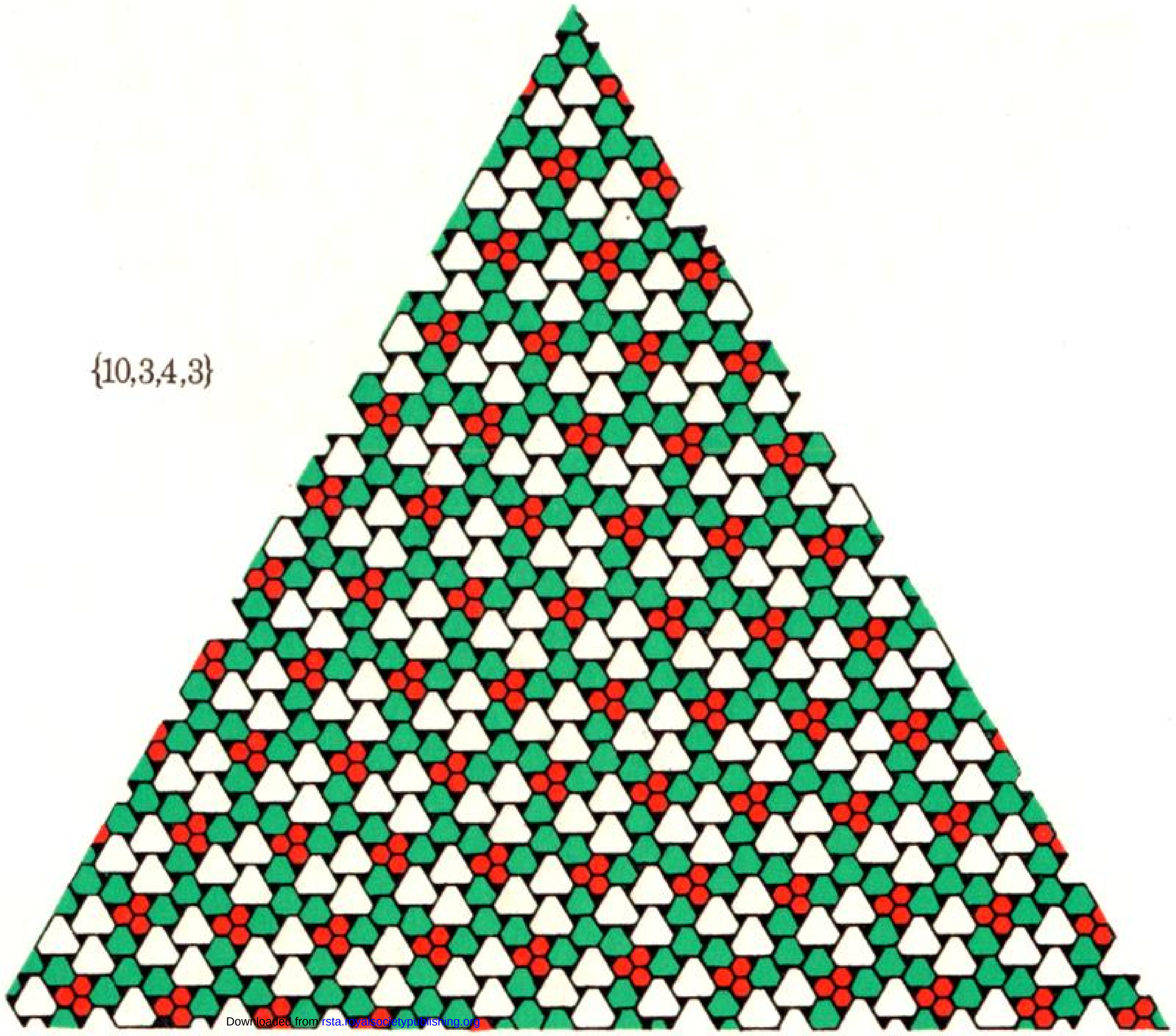


{3,3,1,1}

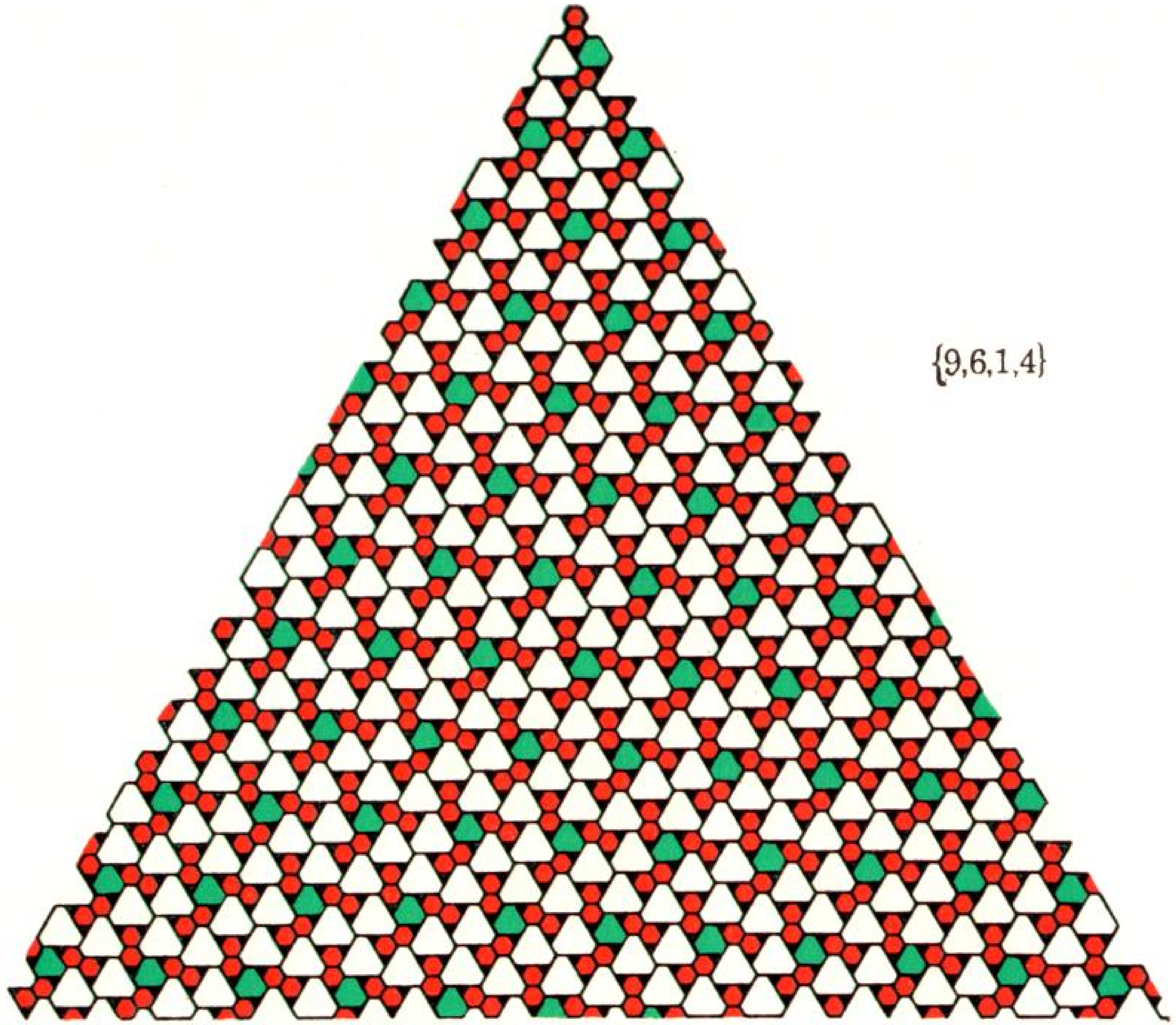


{6,4,0,3}

{10,3,4,3}



{9,6,1,4}



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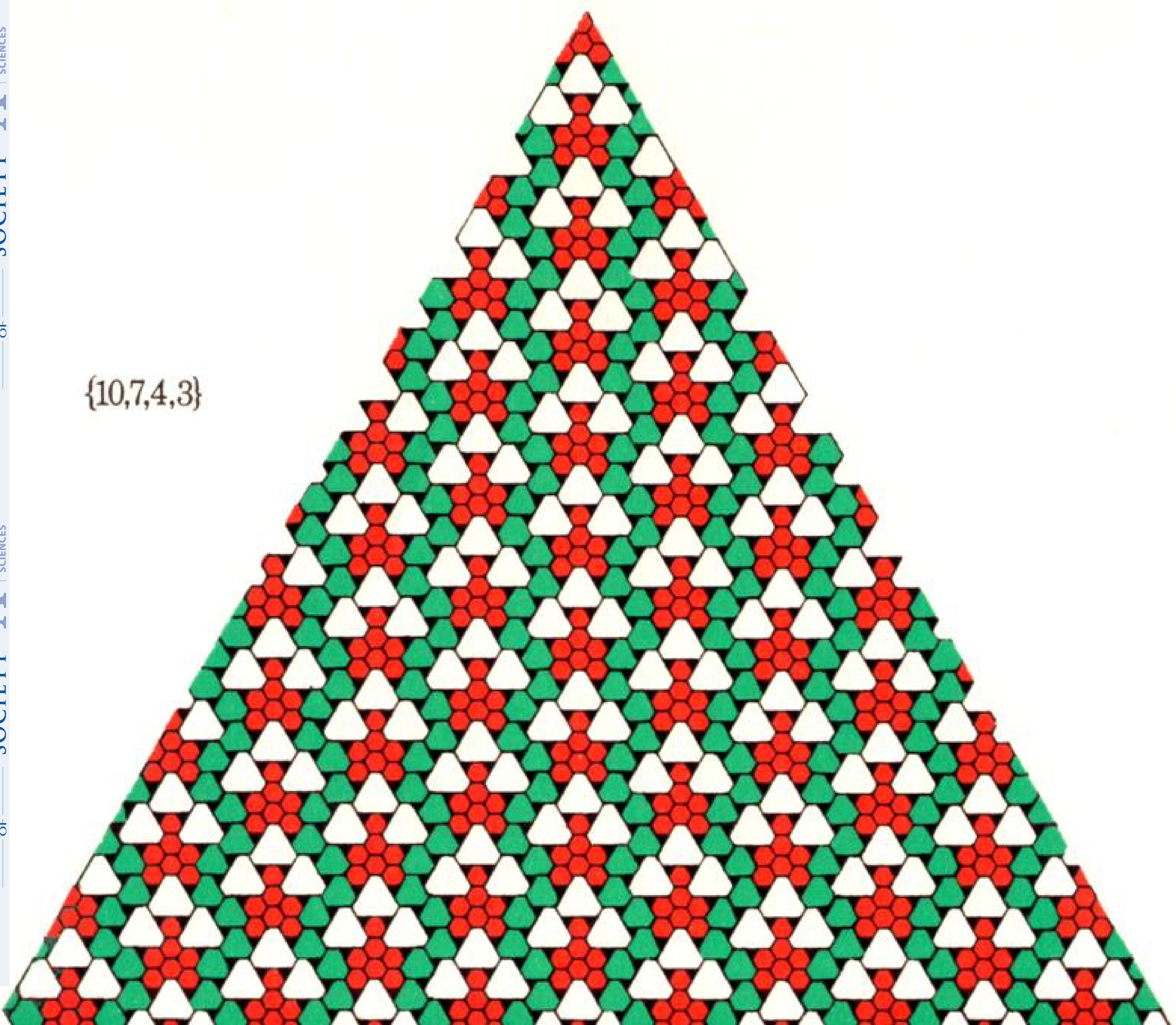
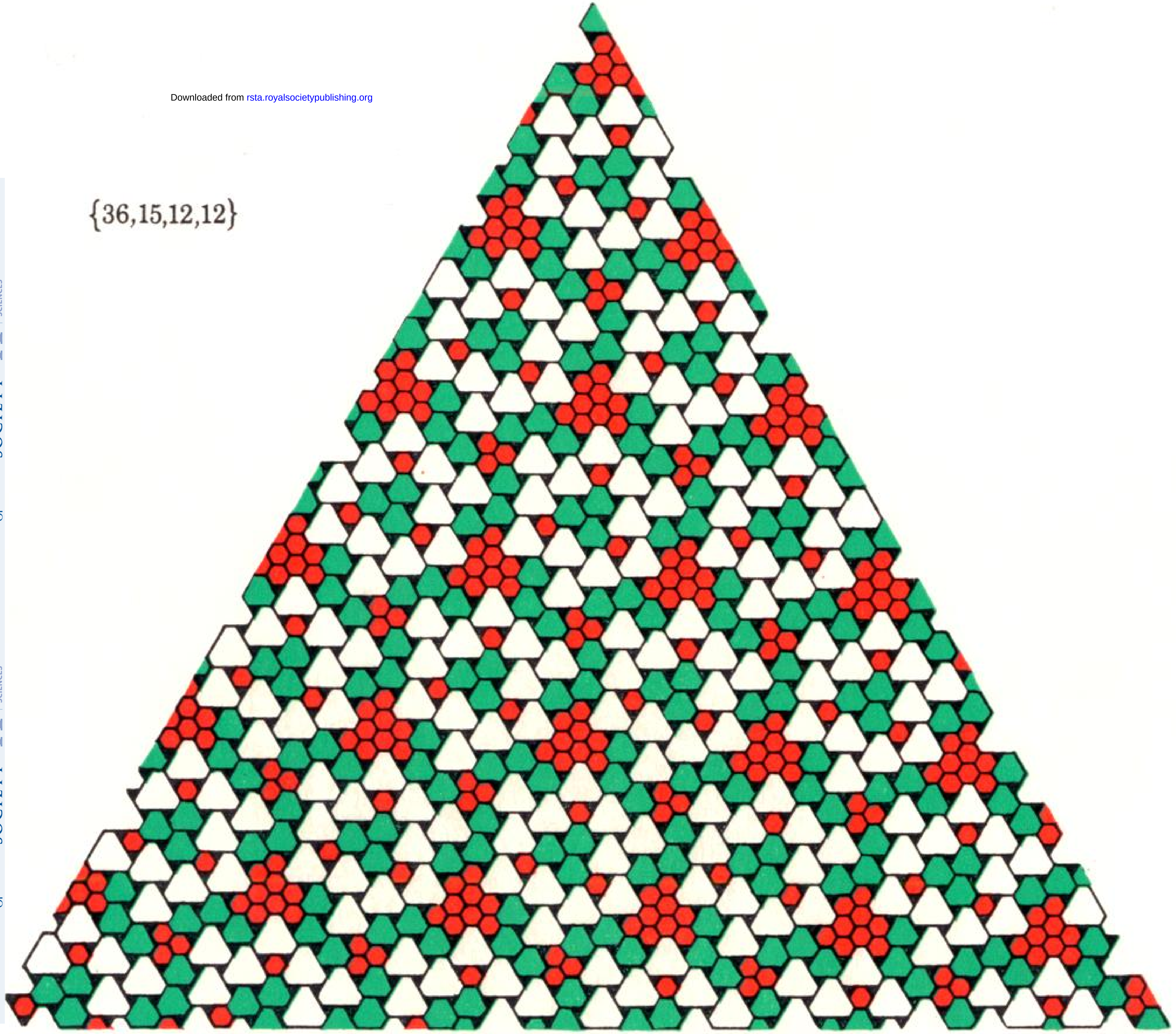


{13,9,7,3}

{36,15,12,12}



{61,21,19,21}